International Journal of Thermal Sciences 49 (2010) 209–217

Contents lists available at ScienceDirect

International Journal of Thermal Sciences

journal homepage: www.elsevier.com/locate/ijts

Frequency-modulated hyperbolic heat transport and effective thermal properties in layered systems

J. Ordóñez-Miranda, J.J. Alvarado-Gil*

Applied Physics Department, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional-Unidad Mérida, Carretera Antigua a Progreso km. 6, A.P. 73 Cordemex, Mérida, Yucatán 97310, México

ARTICLE INFO

Article history:
Received 10 March 2009
Received in revised form
4 June 2009
Accepted 7 July 2009
Available online 29 July 2009

Keywords:
Hyperbolic heat conduction
Cattaneo equation
Effective thermal properties
Layered system

ABSTRACT

In this work heat transport in layered systems is analyzed using a hyperbolic heat conduction equation and considering a modulated heat source for both Dirichlet and Neumann boundary conditions. In the thermally thin case, with Dirichlet boundary condition, the well known effective thermal resistance formula is derived; while for Neumann problem only a heat capacity identity is found, due to the fact that in this case this boundary condition cannot become asymptotically steady when modulation frequency goes to zero. In contrast in the thermally thick regime, heat transport shows a strong enhancement when hyperbolic effects are considered. For this thermal regime, an analytical expression, for both Dirichlet and Neumann conditions, is obtained for the effective thermal diffusivity of the whole system in terms of the thermal properties of the individual layers. It is shown that the magnifying effects on the effective thermal diffusivity are especially remarkable when the thermalization time and the thermal relaxation time are comparable. The limits of applicability of our equation, in the thermally thick regime are shown to provide useful and simple results in the characterization of layered systems. Enhancement in thermal transport and in the effective thermal diffusivity is a direct consequence of having taken into account the fundamental role of the thermal relaxation time in addition to the thermal diffusivity and thermal effusivity of the composing layers. It is shown that our results can be reduced to the ones obtained using Fourier heat diffusion equation, when the thermal relaxation times tend to zero.

1. Introduction

Effective models have provided a useful basis for the interpretation of experimental data and understanding of heat transport in non-homogeneous systems [1]. The most of these models are based on Fourier law, which is supported by an impressive quantity of useful and successful results that show a very good agreement with experimental data for a great variety of experimental conditions [2,3]. However, it is also well known that Fourier heat diffusion law predicts an infinite velocity for heat propagation, in such a way that a temperature change in any part of the material would result in an instantaneous perturbation at each point of the sample. This inconsistency has been studied by different researchers, and a variety of models have been suggested to solve this situation. For a comprehensive account on this subject the reader is referred to the review articles of Joseph and Preziosi [4], Ozisik and Tzou [5] and the recent book by Wang et al. [6]. The origin of this fundamental problem is due to the fact that Fourier law establishes explicitly that, when a temperature gradient at time t is imposed, the heat flux starts instantaneously at the same time *t*. Considering that heat transport is due to microscopic motion and collisions of particles, atoms and molecules, it is straightforward to conclude that the Fourier condition on the velocity of heat transport cannot be sustained [4,7,8]. One of the simplest and accepted models [6] to solve the inconsistency of Fourier law was suggested by Cattaneo [9] and independently by Vernotte [10]. These authors incorporate the finite propagation speed of heat while retaining the basic nature of Fourier law, modifying the heat flux equation in the form:

$$\vec{J}(\vec{x}, t+\tau) = -k\nabla T(\vec{x}, t), \tag{1}$$

where $\vec{J}$ [W/m^2] is the heat flux vector, T [K] is the absolute temperature, k [W/m K] is the thermal conductivity and τ [s] is a thermal property of the medium known as the thermal relaxation time, which represents the time necessary for the initiation of the heat flux after a temperature gradient has been imposed at the boundary of the medium. Eq. (1) establishes that the heat flux does

* Corresponding author. Tel.: +52 999 9429421; fax: +52 999 9875383.
E-mail addresses: eordonez@mda.cinvestav.mx (J. Ordóñez-Miranda), jjag@mda.cinvestav.mx (J.J. Alvarado-Gil).

1290-0729/$ – see front matter © 2009 Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.ijthermalsci.2009.07.005

Nomenclature

c	specific heat, J/kg K
f	frequency, Hz
F	dimensionless parameter
I	light beam intensity, W/m^2
J	heat flux, W/m^2
k	thermal conductivity, W/m K
l	thickness, m
q	complex wave number, m^{-1}
Q	positive constant, W/m^3
R	reflection coefficient
Re()	real part
S	heat source, W/m^3
t	time, *s*
T	temperature, *K*
x	spatial coordinate, m

Greek symbols

α	thermal diffusivity, m^2/s
ε	thermal effusivity, Ws$^{1/2}$/m^2 K
η	efficiency at which the absorbed light is converted into heat
θ	spatial part of the oscillatory temperature, K
Θ	positive constant, K
λ	complex parameter
μ	classical thermal diffusion length, m
ρ	density, kg/m^3
τ	thermal relaxation time, s
χ	dimensionless real parameter
ω	angular frequency, rad/s

Subscripts

ac	relative to the time-dependent temperature
amb	ambient
dc	relative to a time-independent temperature
0	relative to the semi-infinite layer
1	relative to the first finite layer
2	relative to the second finite layer

not start instantaneously, but rather grows gradually with the thermal relaxation time after the application of the temperature gradient. Conversely, τ represents the time necessary for the disappearance of the heat flux after the removal of temperature gradient [4,6].

From Eq. (1), expanding the heat flux vector in Taylor series around $\tau = 0$, and approximating at first order in τ,

$$\vec{J}(\vec{x},t) + \tau\frac{\partial \vec{J}(\vec{x},t)}{\partial t} = -k\nabla T(\vec{x},t). \quad (2)$$

The solution of this equation is given by

$$\vec{J}(\vec{x},t) = -\frac{k}{\tau}e^{-t/\tau}\int_{-\infty}^{t} e^{\xi/\tau}\nabla T(\vec{x},\xi)\mathrm{d}\xi. \quad (3)$$

This equation establishes that the heat flux vector $\vec{J}(\vec{x},t)$ at a certain time t depends on the history of the temperature gradient established in the whole time interval from $-\infty$ to t. This indicates that the heat flux has thermal memory, consequence of the finite value of the thermal relaxation time [11]. In this way, Eq. (3) predicts a dependence of the time path of the temperature gradient rather than an instantaneous response predicted by Fourier law.

Otherwise energy conservation equation is given by [2]

$$\nabla\cdot\vec{J}(\vec{x},t) + \rho c\frac{\partial T(\vec{x},t)}{\partial t} = S(\vec{x},t), \quad (4)$$

where ρ [kg/m^3] is the density, c [J/kg K] is the specific heat of the medium and the source S [W m^3] is the rate per unit volume at which the heat flux is generated. Combining Eqs. (2) and (4), the hyperbolic Cattaneo–Vernotte heat conduction equation is obtained [9,10]

$$\nabla^2 T(\vec{x},t) - \frac{1}{\alpha}\frac{\partial T(\vec{x},t)}{\partial t} - \frac{\tau}{\alpha}\frac{\partial^2 T(\vec{x},t)}{\partial t^2} = -\frac{1}{k}\left(S(\vec{x},t) + \tau\frac{\partial S(\vec{x},t)}{\partial t}\right), \quad (5)$$

where $\alpha = k/\rho c$ is the thermal diffusivity of the medium. On the left hand side of this equation, the second order time derivative term indicates that heat propagates as a wave with a characteristic speed $\sqrt{\alpha/\tau}$. Note that the first order time derivative term corresponds to a diffusive process, which is damping spatially the heat wave. Eq. (5) reduces to the parabolic heat diffusion equation (based on Fourier law) for $\tau \to 0$ or in steady-state conditions $\partial\vec{J}(\vec{x},t)/\partial t = 0$ [4].

The applicability of Cattaneo–Vernotte equation and its generalizations has been widely discussed in the literature [4,6,12–16]. It is clear that a physical system would follow the predicted hyperbolic behavior if the time scale of the heat transport phenomena analyzed is of the order of the thermal relaxation time. This quantity has been reported to be of the order of microseconds (10^{-6} s) to picoseconds (10^{-12} s) for metals, superconductors and semiconductors [7]. These small values of the thermal relaxation time indicate that its effects will not be significant if the physical time scales are of the order of microseconds or larger. In these situations Fourier approach provides adequate results. However, in modern applications such as analysis and processing of materials using ultrashort laser pulses and high speed electronic devices, the finite value of the thermal relaxation time is necessary to be considered [11–17].

One of the most interesting questions is the applicability of the hyperbolic formalism in materials with non-homogeneous inner structure, such as biological tissues and granular materials, in which several authors have claimed that they have observed hyperbolic effects with thermal relaxation times of the order of seconds [12,14]. This has generated a great controversy, because another group of authors have argued that it is enough to consider the traditional Fourier approach [18].

Recently, in the study of heat transport in nanofluids, different research groups have reported thermal conductivities much higher than the values predicted by the conventional mean field models [19,20]. These results have induced to some authors to consider that the hyperbolic equation for heat transfer could be a good option to explain the experimental data for the thermal properties of nanofluids [21]. This is due to the fact that high values of the thermal relaxation times or the presence of nanoelements could generate hyperbolic effects and consequently high thermal conductivity values for a composed system [22], because that in hyperbolic models; heat transport behaves more wave-like than in the traditional Fourier parabolic approach [4].

Fourier [23–28] and hyperbolic models [29–33] have been used to study heat transport in layered systems. These kind of physical systems constitute one of the basic configurations for the analysis and development of effective thermal properties models [1]. This area of research has provided useful and meaningful results for the interpretation of heat diffusion and transport using the Fourier law [24–28]. It is therefore of the main importance to explore the form and consequences that the hyperbolic heat conduction equation could have on the effective thermal properties of layered systems.

In this work, hyperbolic thermal wave transport in two-layer systems is analyzed using the Cattaneo–Vernotte heat conduction equation, considering a modulated heat source. Boundary conditions of Dirichlet and Neumann type are considered. It is shown that when both layers are thermally thin or thermally thick, analytical expressions for the effective thermal properties as a function of the thermal properties of the individual layers can be obtained. The results for the thermally thin layers are the traditional ones obtained using the Fourier approach. In contrast, in the thermally thick regime, it is shown that effective thermal properties also depend on the thermal relaxation time of the composing layers. The consequent enhancement in heat transport is analyzed, in which the most important effects are observed when thermalization time and thermal relaxation time are comparable. In the thermally thick regime, an equation for the effective thermal diffusivity that provides useful and simple results for the characterization of layered systems is derived. The fundamental role of the thermal relaxation time in addition to the thermal diffusivity, thermal conductivity and thermal effusivity of the composing layers is discussed.

2. Formulation of the problem and solutions

The problem to be solved is to find the effective thermal conductivity and diffusivity of the layered systems shown in Fig. 1, using Cattaneo–Vernotte approach (see Eq. (5)). The system may be excited at the surface $x = 0$ by a modulated temperature $T(x = 0, t)$ or heat source $S(x = 0, t)$ at frequency f of the form [34,35]:

$$T(x = 0, t) = \Theta(1 + \cos(\omega t)) = \mathrm{Re}\left[\Theta\left(1 + e^{i\omega t}\right)\right], \tag{6a}$$

$$S(x = 0, t) = Q(1 + \cos(\omega t)) = \mathrm{Re}\left[Q\left(1 + e^{i\omega t}\right)\right], \tag{6b}$$

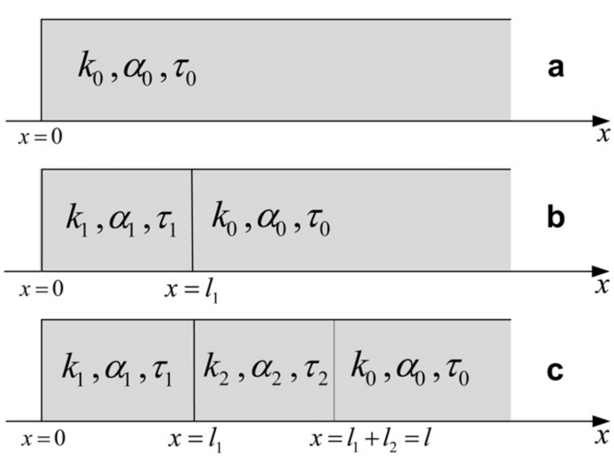

Fig. 1. Schematic diagram of the studied layered systems. (a) Semi-infinite one-dimensional sample of thermal conductivity k_0, thermal diffusivity α_0 and thermal relaxation time τ_0. (b) A layer of conductivity k_1, diffusivity α_1, relaxation time τ_1 and thickness l_1 is added to the semi-infinite system of (a). (c) An additional layer of conductivity k_2, diffusivity α_2, relaxation time τ_2 and thickness l_2 is inserted between the two layers of the previous system.

where $\omega = 2\pi f$, $\mathrm{Re}(\xi)$ is the real part of ξ and Θ [K] and Q [W/m^3] are two positive constants. For any of these thermal excitations, the temperature at any point inside the sample ($x \geq 0$) is given by:

$$T(x, t) = T_{\mathrm{amb}} + T_{\mathrm{dc}}(x) + T_{\mathrm{ac}}(x, t), \tag{7}$$

with T_{amb} being the ambient temperature. T_{dc} and $T_{\mathrm{ac}}(x,t) = \mathrm{Re}[\theta(x)e^{i\omega t}]$ are the stationary raising and periodic components of the temperature, due to the first and second terms of the heat source, respectively. From now on, the operator Re() will be omitted, taking into account the convention that the real part of expressions must be taken to obtain physical quantities. Our attention will be focused on the oscillatory part of the temperature, due to the fact that it is the quantity of interest in lock-in and similar detection techniques.

Inserting Eq. (7) into Eq. (5), in its one-dimensional form, considering that there are not any internal thermal excitations and taking into account that x and t are mutually independent variables, then for $x \geq 0$ it is obtained that $\theta(x)$ satisfies the following differential equation:

$$\frac{d^2\theta(x)}{dx^2} - \frac{i\omega}{\alpha}(1 + i\omega\tau)\theta(x) = 0, \tag{8}$$

whose general solution is

$$\theta(x) = Ae^{qx} + Be^{-qx}, \tag{9}$$

where A and B are two constants that depend on the boundary conditions at $x = 0,\ l_1,\ l$ of the corresponding problem and q is given by:

$$q = \sqrt{\frac{i\omega}{\alpha}}\sqrt{1 + i\omega\tau} = \frac{\chi + i\chi^{-1}}{\mu}, \tag{10a}$$

$$\mu = \sqrt{\frac{2\alpha}{\omega}} = \sqrt{\frac{\alpha}{\pi f}}, \tag{10b}$$

$$\chi = \sqrt{\sqrt{1 + (\omega\tau)^2} - \omega\tau}. \tag{10c}$$

Assuming that the layers are in perfect thermal contact [36,37], the boundary conditions obtained from the usual requirement of temperature and heat flux continuity at the interfaces are given by:

$$\theta\left(x^-\right) = \theta\left(x^+\right), \tag{11a}$$

$$\frac{k(x^-)}{1 + i\omega\tau^-}\frac{d\theta(x^-)}{dx} = \frac{k(x^+)}{1 + i\omega\tau^+}\frac{d\theta(x^+)}{dx}, \tag{11b}$$

where the superscripts "+" and "−" indicate that the limit $x \to l_1 (x \to l)$ is taken from the right and left of the length $l_1(l)$, respectively. The form of Eq. (11b) may be derived by either of Eqs. (2) or (3). In what follows, the explicit solutions established by the two types of thermal excitations given in Eqs. (6a) and (6b) are considered, the first one specifying the temperature and the other heat flux, in both cases, at the surface $x = 0$.

2.1. Dirichlet problem

In this case, according to Eq. (6a); the following boundary condition is considered:

$$\theta(x = 0) = \Theta, \tag{12}$$

where Θ is a non-zero positive constant.

2.1.1. Semi-infinite layer (Fig. 1a)

In this case the physically acceptable solution [$\theta(x) \to 0$ when $x \to \infty$] is given by:

$$\theta(x) = \Theta e^{-q_0 x}, \tag{13}$$

where q_0 is defined by Eqs. (10a)–(10c) with the thermal properties of the semi-infinite layer, and $x \geq 0$.

2.1.2. Semi-infinite layer in contact with the layer of thickness l_1 (Fig. 1b)

In this case, the solution for the spatial part of the thermal wave and for $x \geq l_1$ is given by:

$$\theta(x) = \frac{2\Theta}{(1+\lambda_{01})e^{q_1 l_1} + (1-\lambda_{01})e^{-q_1 l_1}} e^{-q_0(x-l_1)}, \tag{14}$$

where q_1 is defined by Eqs. (10a)–(10c) for the thermal properties of the first layer (layer 1), and $\lambda_{01} = \frac{k_0 q_0}{k_1 q_1}\frac{1+i\omega\tau_1}{1+i\omega\tau_0}$. Since $q_1 l_1 = \frac{l_1}{\mu_1}(\chi_1 + i\chi_1^{-1})$, the solutions given by Eq. (14) can be classified as thermally thin when $l_1\chi_1^{-1}/\mu_1 \ll 1$ (which implies that $l_1\chi_1/\mu_1 \ll 1$, because $\chi_1 \leq 1$) and thermally thick for a layer such that $l_1\chi_1/\mu_1 \gg 1$ (which implies that $l_1\chi^{-1}/\mu_1 >> 1$, because $\chi^{-1} \geq 1$). This classification will be crucial in the comprehension and analyses of hyperbolic heat transport phenomena, providing useful and convenient approximations of Eq. (14).

- For a thermally thin material ($l_1\chi_1^{-1}/\mu_1 \ll 1$), Eq. (14) takes the form,

$$\theta(x) \approx \Theta e^{-k_0 q_0 \frac{l_1}{k_1}} e^{-q_0(x-l_1)}. \tag{15}$$

- In the case of a thermally thick material ($l_1\chi_1/\mu_1 \gg 1$), Eq. (14) reduces to,

$$\theta(x) \approx \Theta\left(\frac{2}{1+\lambda_{01}}\right) e^{-q_1 l_1} e^{-q_0(x-l_1)}. \tag{16}$$

Evaluating Eq. (16) at $x = l_1$ it is obtained that $\theta(l_1) \approx \Theta(2/(1+\lambda_{01}))e^{-q_1 l_1}$. Comparing this equation with Eq. (13), for a semi-infinite medium, it can be observed that they are similar and only differ by the factor $2/(1+\lambda_{01})$. This interface term is a consequence of the insertion of layer 1 in contact with the semi-infinite layer.

2.1.3. Two finite layers in contact with a semi-infinite layer (Fig. 1c)

In this case for $x \geq l$, it can be shown that,

$$\theta(x) = \frac{4\Theta\eta_1\eta_2}{D} e^{-q_0(x-l)}, \tag{17}$$

where,

$$D = \left(\eta_1^2 + 1\right)\beta_1 + \lambda_{01}\left(\eta_1^2 - 1\right)\beta_2. \tag{18a}$$

$$\beta_1 = \left(\eta_2^2 + 1\right) + \lambda_{02}\left(\eta_2^2 - 1\right). \tag{18b}$$

$$\beta_2 = \left(\eta_2^2 + 1\right) + \lambda_{20}\left(\eta_2^2 - 1\right). \tag{18c}$$

$$\eta_j = e^{q_j l_j}, \quad j = 1, 2. \tag{18d}$$

$$\lambda_{mn} = \frac{k_m q_m}{k_n q_n}\frac{1+i\omega\tau_n}{1+i\omega\tau_m}, \quad m, n = 0, 1, 2. \tag{18e}$$

In analogy with the previous system, the approximations of Eq. (17) for thermally thin and thermally thick materials can be made, as follows:

- For thermally thin materials ($l_j\chi_j^{-1}/\mu_j \ll 1$; $j = 1, 2$), Eq. (17) can be written as,

$$\theta(x) \approx \Theta e^{-k_0 q_0\left(\frac{l_1}{k_1} + \frac{l_2}{k_2}\right)} e^{-q_0(x-l_1)}. \tag{19}$$

If it is considered that the two finite layers of thicknesses l_1 and l_2 behave as if they were a single one of thickness $l = l_1 + l_2$ and effective thermal conductivity k, according to Eq. (15) the temperature in this layer can be written in the form:

$$\theta(x) \approx \Theta e^{-k_0 q_0 \frac{l}{k}} e^{-q_0(x-l_1)}. \tag{20}$$

Comparing Eqs. (19) and (20), it is obtained that:

$$\frac{l}{k} = \frac{l_1}{k_1} + \frac{l_2}{k_2}, \tag{21}$$

which has been previously obtained for stationary systems [2], it is widely used in thermal characterization [24–27,38] and it has been shown to be valid for the same boundary conditions in parabolic heat transport [26,39]. In our case, Eq. (21) is obtained using the hyperbolic model, and for the case in which layers 1 and 2 are thermally thin, as defined previously in the form $l_j\chi_j^{-1}/\mu_j \ll 1$, for $j = 1, 2$. Note that if this condition is solved for ω, the following inequality is obtained,

$$\omega << \frac{\omega_j}{\sqrt{1 + 2\omega_j\tau_j}}, \tag{22}$$

where $\omega_j = 2\alpha_j/l_j^2$ is the classical cutoff frequency of the layer $j = 1, 2$ [3].

- For thermally thick materials ($l_j\chi_j/\mu_j >> 1$; $j = 1, 2$), Eq. (17) is given by,

$$\theta(x) \approx \Theta\left(\frac{2}{1+\lambda_{02}}\right)\left(\frac{2}{1+\lambda_{21}}\right) e^{-(q_1 l_1 + q_2 l_2)} e^{-q_0(x-l_1)}. \tag{23}$$

Considering again that the two finite layers of thicknesses l_1 and l_2 behave as if they were a single effective layer of thickness $l = l_1 + l_2$ and effective thermal diffusivity α, according to Eq. (16) and the remarks made after it, the temperature in this layer can be written in the form:

$$\theta(x) \approx \Theta\left(\frac{2}{1+\lambda_{02}}\right) e^{-ql} e^{-q_0(x-l)}. \tag{24}$$

Comparing Eqs. (23) and (24), it can be shown that:

$$\chi\frac{l}{\sqrt{\alpha}} = \chi_1\frac{l_1}{\sqrt{\alpha_1}} + \chi_2\frac{l_2}{\sqrt{\alpha_2}} + \sqrt{\frac{2}{\omega}}\ln\left[\frac{1}{2}|1+\lambda_{21}|\right], \tag{25}$$

where,

$$\lambda_{21} = \frac{\varepsilon_2}{\varepsilon_1}\sqrt{\frac{1+i\omega\tau_1}{1+i\omega\tau_2}} = \frac{\varepsilon_2}{\varepsilon_1}\sqrt{\frac{1+(\omega\tau_1)^2}{1+(\omega\tau_2)^2}} e^{\frac{i}{2}\arctan\left(\frac{(\tau_1-\tau_2)\omega}{1+\tau_1\tau_2\omega^2}\right)}, \tag{26}$$

with $\varepsilon_j = \sqrt{k_j\rho_j c_j}$ for $j = 1, 2$; being the thermal effusivity of the j-th layer, χ is given by Eq. (10c) for the effective thermal relaxation time τ of the two-layer system. According to the analysis of the collision term in Boltzmann transport equation for the non-equilibrium statistical mechanics, the following relation could be obtained [40]:

$$\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2}. \quad (27)$$

It is important to emphasize that Eq. (25) is valid for thermally thick layers, for which the frequency must follow the inequality:

$$\omega >> \frac{\omega_j}{\sqrt{1 - 2\omega_j \tau_j}}, \quad (28)$$

with $\omega_j = 2\alpha_j / l_j^2$. In order to understand the consequences of Eq. (25), different limiting cases are considered:

- If the thermal relaxation times are zero $\tau_1 = \tau_2 = \tau = 0$ this is the parabolic limit of Cattaneo–Vernotte equation, in this case Eq. (25) for the effective thermal diffusivity reduces to:

$$\frac{l}{\sqrt{\alpha}} = \frac{l_1}{\sqrt{\alpha_1}} + \frac{l_2}{\sqrt{\alpha_2}} + \sqrt{\frac{2}{\omega}} \ln\left[\frac{1}{2}\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\right)\right]. \quad (29)$$

This formula was derived under the same Dirichlet boundary condition, using Fourier law, by Lucio et al. [26] and it reduces to a previous one derived by Tominaga and Ito [24] in the limit $\omega \to \infty$. Lucio et al. [26] have shown that the ω dependent term is necessary to explain previously reported experimental data [25,38], for a frequency range in which the individual layers are thermally thick in the traditional definition ($l_j/\mu_j \gg 1$) of Rosencwaig theory [35]. This indicates that the thermal regime where Eq. (29) is valid, corresponds to frequencies which are not high enough. As a consequence it is sufficient to consider parabolic effects. Therefore Eq. (25) is the generalization in the hyperbolic approach, of the effective thermal diffusivity result obtained by Lucio et al. [26].

- Considering that the frequency is not very high, in such a way that the thermal diffusivities and thermal relaxation times of the individual layers fulfill the conditions $\omega_j \tau_j / \sqrt{1 - 2\omega_j \tau_j} \ll \omega \tau_j \ll 1$ for $j = 1, 2$ and performing a first order approximation in $\omega \tau_j$, it can be shown that Eq. (25) can be written as:

$$\frac{l}{\sqrt{\alpha}}(1 - \pi f \tau) = \frac{l_1}{\sqrt{\alpha_1}}(1 - \pi f \tau_1) + \frac{l_2}{\sqrt{\alpha_2}}(1 - \pi f \tau_2) + \frac{1}{\sqrt{\pi f}} \ln\left[\frac{1}{2}\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\right)\right]. \quad (30)$$

This expression is similar to that obtained for the parabolic approximation (see Eq. (29)); in fact the logarithmic part is the same. However, the effective thermal diffusivity is slightly different, due to the weak dependence on the thermal relaxation times.

- For high modulation frequencies in which the condition $\omega \tau_j \gg \max\{1, \omega_j \tau_j / \sqrt{1 - 2\omega_j \tau_j}\}$ being $j = 1, 2$ is fulfilled, and for an approximation of first order in $(\omega \tau_j)^{-1}$, the following expression is obtained from Eq. (25):

$$\frac{l}{\sqrt{\alpha \tau}} = \frac{l_1}{\sqrt{\alpha_1 \tau_1}} + \frac{l_2}{\sqrt{\alpha_2 \tau_2}} + 2\ln\left[\frac{1}{2}\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\sqrt{\frac{\tau_1}{\tau_2}}\right)\right], \quad (31)$$

which is independent of the modulation frequency, t is only valid for very high frequencies and it represents the hyperbolic generalization of the Tominaga and Ito formula [24,28].

From the definition of thermally thick layers (see Eq. (28)), it is important to note that there is a constraint among the thermal relaxation time, the thermal diffusivity and the thickness of the layers, given by:

$$\tau_j < \frac{l_j^2}{4\alpha_j} \text{ or } l_j > 2\sqrt{\alpha_j \tau_j}. \quad (32)$$

The first inequality indicates that for a layer of thermal diffusivity α_j and thickness l_j, its thermal relaxation time has a least upper bound $l_j^2/4\alpha_j$. This quantity is proportional and very close to the thermalization time, obtained in the analysis of thermal transients [3]. The second inequality indicates that when the thermal diffusivity and thermal relaxation time of a layer are known, its thickness cannot be made arbitrarily small. This is a limit for the validity of the hyperbolic approach in the analysis of thermal transport and is closely connected with the consideration of heat transport as a collective motion.

2.2. Neumann problem

Considering that the surface $x = 0$ is excited by a periodic heat flux, this situation can be fulfilled when the opaque surface of a material is uniformly illuminated by a laser light beam of periodically modulated intensity; the heat source is given by [11,28]:

$$I_0[1 + \cos(\omega t)]/2 = \mathrm{Re}\left[I_0\left(1 + e^{i\omega t}\right)/2\right],$$

where the $I_0 = F\eta(1 - R)I$, being F a parameter determined by the optical, thermal and geometric properties of the first layer, η is the efficiency at which the absorbed light is converted into heat, R is the reflection coefficient of the surface at $x = 0$ and I [W/m^2] is the intensity of the light beam. Considering that the sample is uniformly illuminated with a fixed light source, the factor I_0 can be taken as nearly constant and independent of the modulation frequency as it is usually assumed in similar problems [11,28,41]. The boundary condition in this case has the following form:

$$-\frac{k}{1 + i\omega\tau} \frac{d\theta(x)}{dx}\bigg|_{x=0} = \frac{I_0}{2}, \quad (33)$$

where $k(= k_0 \text{ or } k_1)$ is the thermal conductivity and $\tau(= \tau_0 \text{ or } \tau_1)$ is the thermal relaxation time, both of the first layer. From Eqs. (9), (11) and (33) and for the three systems shown in Fig. 1, the following results are obtained.

2.2.1. Semi-infinite layer (Fig. 1a)

In this case for $x \geq 0$, the temperature is given by:

$$\theta(x) = \frac{I_0}{2} \frac{1 + i\omega\tau_0}{k_0 q_0} e^{-q_0 x}. \quad (34)$$

2.2.2. Finite layer in contact with a semi-infinite layer (Fig. 1b)

In this case for $x \geq l_1$, it is obtained that:

$$\theta(x) = \frac{I_0}{2} \frac{1 + i\omega\tau_1}{k_1 q_1} \frac{2}{(1 + \lambda_{01})e^{q_1 l_1} - (1 - \lambda_{01})e^{-q_1 l_1}} e^{-q_0(x - l_1)}. \quad (35)$$

2.2.3. Two finite layers in contact with a semi-infinite layer (Fig. 1c)

It can be shown that for $x \geq l$, the expression for the temperature is:

$$\theta(x) = \frac{I_0}{2} \frac{1 + i\omega\tau_1}{k_1 q_1} \frac{4\eta_1 \eta_2}{N} e^{-q_0(x - l)}, \quad (36)$$

where

$$N = \left(\eta_1^2 - 1\right)\beta_1 + \lambda_{01}\left(\eta_1^2 + 1\right)\beta_2, \quad (37)$$

and all the other parameters have been defined previously (see Eqs. (18b)–(18e)). Following a similar procedure for the analysis of the solutions obtained with the Neumann boundary condition than the one used in the Dirichlet problem, and comparing the temperature of a system of one finite layer (see Eq. (35)) with the temperature of the system for two finite layers (see Eq. (36)), the following results are obtained:

- If both finite layers are thermally thin, it can be shown that

$$\rho c l = \rho_1 c_1 l_1 + \rho_2 c_2 l_2. \tag{38}$$

Since the quantity ρclA with A being the transversal area of the layers, represents the heat capacity of the effective layer, Eq. (38) is just an expected identity, due to the fact that this property is an extensive thermodynamic variable. This result has been obtained previously based on parabolic theory for conventional thermal wave phenomena under the same Neumann boundary condition [27].

For thermally thin composing layers, ruled by hyperbolic or parabolic models and obeying the Neumann boundary condition, it is not possible to obtain a useful formula for the effective thermal properties of a layered system. This result is in strong contrast with Eq. (21) for the effective thermal conductivity obtained using the Dirichlet boundary conditions in the parabolic and hyperbolic approaches and for stationary heat conduction. Given the characteristics of the boundary conditions, in the limit when the frequency goes to zero, Neumann condition implies an asymptotically increasing deposition of thermal energy on the sample surface. This is not consistent with steady state conditions that guarantee the validity of the equation for the effective thermal conductivity given by Eq. (21). Therefore the heat capacity identity obtained in Eq. (38) for Neumann boundary condition is only a consistency equation that must be expected to be fulfilled.

- If both composing layers are thermally thick, for the Neumann boundary condition the same formula found in Dirichlet problem (see Eq. (25)) is obtained. This indicates that in an experiment dedicated to measure the effective thermal properties, for high modulation frequencies, it is equivalent to establish a temperature or a heat flux boundary condition as the excitation source at the surface of the sample.

3. Results and discussions

In this section, the consequences of the obtained equations for the effective thermal properties are explored comparing with the conventional parabolic results. Typical values for thermal diffusivities, thickness and thermal relaxation times reported in the literature for crystal solids are used (see Table 1) [4,7]. In spite of the small size of the thermal relaxation time used here, it can be shown that for $\tau \neq 0$, performing modulation frequency scans, a region in which the hyperbolic heat conduction equation and its consequences are dominant can always be found. Therefore our analyses will be useful for a very wide range of thermal parameters. On the other hand, since for thermally thin layers, parabolic and hyperbolic models provide the same results under the Dirichlet boundary condition (see Eq. (21)) and an expected identity for Neumann problem (see Eq. (38)), only the case in which both composing layers are thermally thick is going to be analyzed.

Table 1
Thicknesses, thermal diffusivities and thermal relaxation times of the composing layers.

l_1 (μm)	l_2 (μm)	α_1 (mm^2/s)	α_2 (mm^2/s)	τ_1 (ηs)	τ_2 (ηs)
20	25	50	20	1	5

In Fig. 2 the effective thermal diffusivity predicted by the parabolic model (see Eq. (29)) as a function of the modulation frequency is presented, for various values of the ratio of the thermal effusivity of layer 2 divided by the thermal effusivity of layer 1 ($\varepsilon_2/\varepsilon_1$). In this case both composing layers forming the system are thermally thick for frequencies such that $f \gg 4 \times 10^4$ Hz, in which Eq. (29) is valid.

In Fig. 2, it can be observed that the effective thermal diffusivity is always smaller than the larger thermal diffusivity of the composing layers, being maximum when the first layer is a perfect thermal conductor ($\varepsilon_2/\varepsilon_1 = 0$) and minimum when the first layer is a perfect thermal insulator ($\varepsilon_2/\varepsilon_1 \gg 1$). It is important to note that in the parabolic limit, for very high frequencies, the curves of effective thermal diffusivity converge to the horizontal limit line ($\varepsilon_1 = \varepsilon_2$), with thermal diffusivity = 28.57 mm^2/s. This result also corresponds to the predicted by Tominaga and Ito formula [24,28].

In Fig. 3, the hyperbolic effective thermal diffusivity (see Eq. (25)) as a function of the modulation frequency is shown, for the thermal properties given in Table 1 and various values of the ratio of thermal diffusivity of layer 2 divided by the thermal effusivity of layer 1 ($\varepsilon_2/\varepsilon_1$).

In Fig. 3, it is shown that when the product $\omega\tau << 1$ ($f << 3.18 \times 10^7$ Hz), the behavior of the hyperbolic effective thermal diffusivity is very similar to the one predicted by the parabolic model. In contrast for $\omega\tau \gg 1$, the effective thermal diffusivity grows with modulation frequency, in such a way that surpasses the value of the thermal diffusivity of layer 1. For even larger values of the modulation frequency, thermal diffusivity becomes independent of the modulation frequency. This corresponds to the behavior predicted by Eq. (31). It is important to emphasize that the values of the effective thermal diffusivity are larger than the ones obtained using the parabolic model.

In Fig. 4, the effective thermal diffusivities predicted by the parabolic (dashed line) and hyperbolic (solid lines) models for a system formed by two thermally thick layers as a function of the modulation frequency are represented. The ratio of thermal diffusivities is $\varepsilon_2/\varepsilon_1 = 1/2$ and three different values of the thermal

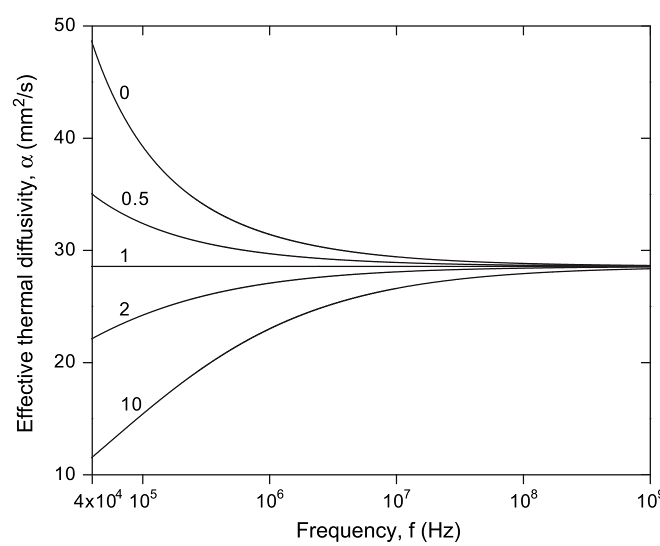

Fig. 2. Effective thermal diffusivity predicted by the parabolic model for a system of thermally thick layers as a function of the modulation frequency, with different values of the ratio $\varepsilon_2/\varepsilon_1$ of thermal effusivities of the composing layers.

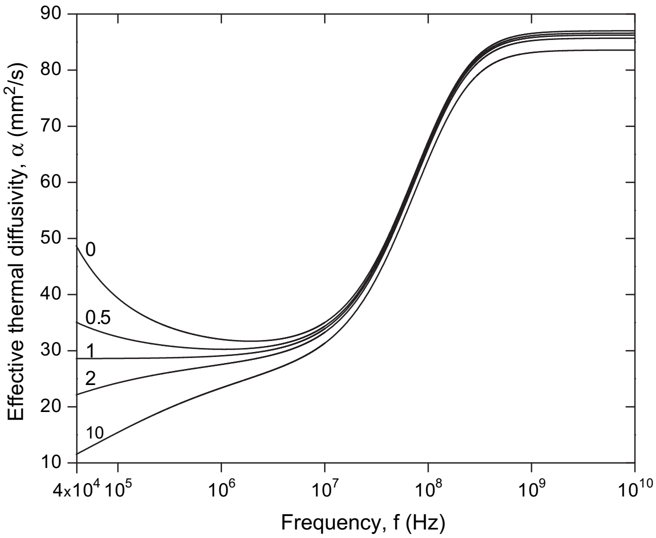

Fig. 3. Effective thermal diffusivity predicted by the hyperbolic approach when both composing layers are thermally thick as a function of the modulation frequency, for different values of the ratio of thermal effusivities ($\varepsilon_2/\varepsilon_1$) and the thermal properties shown in Table 1.

relaxation time of layer 2 are used. It can be observed that the thermal relaxation time has a strong influence on the high values of the effective thermal diffusivity. In fact, higher thermal relaxation time produces an increment of the effective thermal diffusivity at high frequencies. It is from these results that the unusually high values for thermal diffusivity can be understood. As expected from the Cattaneo–Vernotte equation, hyperbolic effects are more notorious for high thermal relaxation times. This produces a more efficient heat transport due to the increased importance of the second derivative in the Cattaneo–Vernotte equation, raising the wave-like behavior with respect to the diffusive contribution. It is important to emphasize that the analysis in terms of effective thermal diffusion privileges the role of thermal diffusivity above the thermal relaxation time. Taking into account that thermal diffusivity measures the spatial decay of the thermal wave; and given that this decay is smaller when hyperbolic effects are dominant, this has the effect of appearing as an increase of the effective thermal diffusivity.

As it was expected from Eq. (31), Fig. 4 also shows that at very high frequencies, the effective thermal diffusivity reaches a maximum value and stays stable for even higher frequencies. This indicates that the decay of the amplitude of the hyperbolic thermal waves has a maximum limit as a function of the frequency, due to the fact that at very high frequencies the dominant term in Cattaneo–Vernotte equation is the corresponding to the second derivative that determines a dominant wave-like behavior.

These results indicate that the unusual observed values of effective thermal properties in composed systems can be due to the fact that heat transport could be strongly influenced by hyperbolic effects consequence of relatively high values of thermal relaxation times. These effects would be especially important in systems with heterogeneous complex structure where the presence of nanoelements could induce the appearance of dual phase lagging heat conduction as reported by several authors [22,32,42–44]. This area of research deserves more investigation, mainly in the development of effective models for complex geometries of heterogeneous systems [1,45–49].

The comparison of the temperature profile obtained directly from the Cattaneo–Vernotte equation for the two-layer system (see Eq. (23)) with the temperature profile for a one layer system with an effective thermal diffusivity given by Eq. (24) is shown in Fig. 5. In this figure the normalized amplitude ($|\theta(x = l)|/\Theta$) of the spatial part of the temperature as a function of the modulation frequency is represented. The values of the thickness and thermal diffusivities of the individual layers were taken from Table 1. The thermal relaxation times are $\tau_1 = 1 \times 10^{-6}$ s and $\tau_2 = 7 \times 10^{-6}$ s for each layer, which were taken very close to their corresponding least upper bounds ($\tau_1 = 2 \times 10^{-6}$ s, $\tau_2 = 7.8 \times 10^{-6}$ s), given by Eq. (32). Calculations have been made considering that the ratio of thermal effusivities of the composing finite layers is $\varepsilon_2/\varepsilon_1 = 1/2$, $\varepsilon_0/\varepsilon_2 = 4/5$ and $\tau_0 = 3 \times 10^{-9}$ s. In this case for frequencies such that $f \gg 5 \times 10^4$ Hz, both layers are thermally thick and Eqs. (23) and (24) are valid.

Fig. 5 shows that,

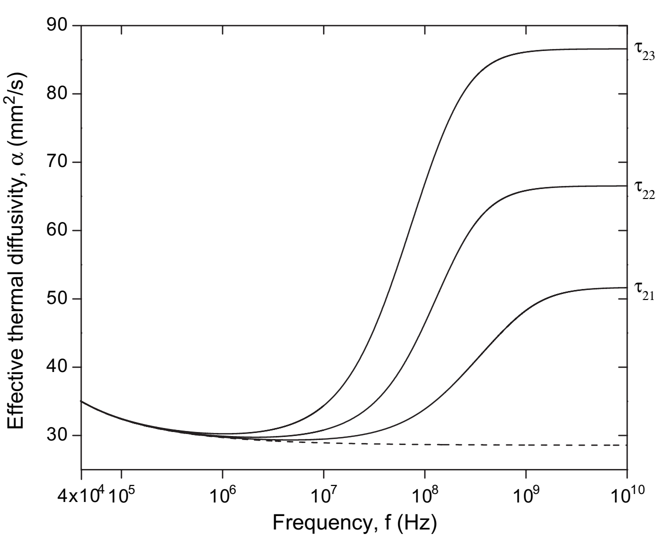

Fig. 4. Frequency dependence of the effective thermal diffusivity predicted by the parabolic (dashed line) and hyperbolic (solid lines) models for a layered system in which both finite layers are thermally thick, when the ratio of thermal effusivities is $\varepsilon_2/\varepsilon_1=1/2$, with $\tau_1=1 \times 10^{-9}$ s and τ_2 takes the values: $\tau_{21} = 2 \times 10^{-10}$ s, $\tau_{22} = 2 \times 10^{-9}$ s and $\tau_{23} = 5 \times 10^{-9}$ s.

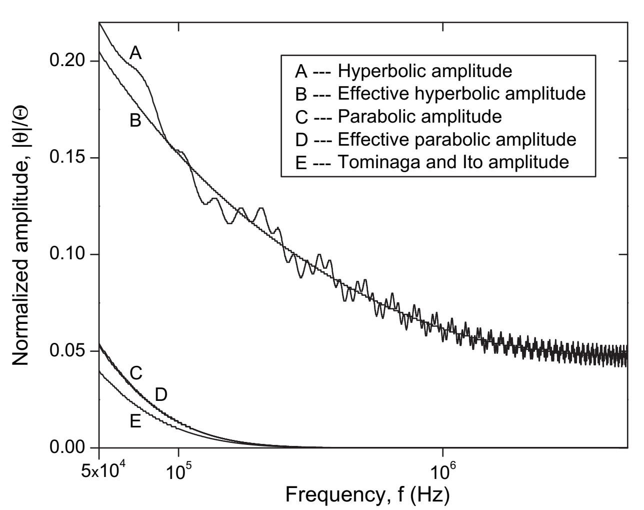

Fig. 5. Frequency dependence of the normalized amplitude of the temperature, calculated directly using the parabolic and hyperbolic equations for a two-layer system. The results for the amplitude using hyperbolic and parabolic effective models for thermally thick composing layers systems are also presented. The parabolic amplitude and the effective parabolic one are practically superimposed.

- The amplitude of the temperature predicted by Cattaneo–Vernotte equation for a two-layer system differs appreciably from the amplitude predicted by Fourier law and effective models derived from it. This could be expected, given that the wave-like form of Cattaneo–Vernotte equation permits an enhancement of the heat transfer as compared with the purely diffusive parabolic behavior predicted by Fourier equation. This enhancement is remarkable when the modulation frequency is much larger than $\omega_j = 2\alpha_j/l_j^2$ or equivalently when the thermal relaxation time of each layer is close to its thermalization time ($l_j^2/4\alpha_j$). In this case the thermal diffusivity is given by Eq. (31).
- It can be observed that the effective model developed in this work provides a good approximation for the temperature profile calculated directly from Cattaneo–Vernotte equation for a two-layer system. This approximation improves when the modulation frequency increases. In the case shown in Fig. 5, this happens when the frequency is such that $f \gg 5 \times 10^4$ Hz. It can be inferred that the effective thermal diffusivity formula provides a useful result in the study of layered systems and may be the basis for the development of effective thermal properties formulas in the study of complex heterogeneous systems. This could be helpful in the evaluation of the role of hyperbolic effects in systems where abnormally high thermal conductivities have been reported, in contradiction with the values predicted by the traditional mean field theories based on Fourier law [19,20].
- For higher frequencies ($\omega\tau_j \gg 1,\ j = 1, 2$), in the hyperbolic model the amplitude becomes independent of the modulation frequency while in the parabolic model it tends to zero, which indicates that for these range of frequencies the hyperbolic thermal waves can travel larger distances than the predicted by the parabolic model.

It is straightforward to show that when the thermal relaxation times are changed, restricted to values allowed by Eq. (32), the agreement between the approximated amplitude using the corresponding effective model (see Eq. (24)) and the amplitude calculated directly using the Cattaneo–Vernotte equation (see Eq. (23)), remains. Additionally the difference between the results obtained from the parabolic and hyperbolic approaches, decreases when the thermal relaxation time of each layer moves away of their corresponding least upper bounds and becomes negligible when the thermal relaxation times of each layer tend to zero.

The presented results for the effective thermal properties can be used as the basis in the development of general formulas for the analysis of thermal properties when considering thermal resistance between adjacent layers, more general approaches as dual phase lag models or more sophisticated ones using Boltzmann transport equation [16,32,42–44].

4. Conclusions

Hyperbolic thermal wave transport in layered systems is analyzed using the Cattaneo–Vernotte heat conduction equation considering a modulated thermal excitation with Dirichlet and Neumann boundary conditions. It has been shown that when both layers are thermally thin or thermally thick, analytical expressions for the effective thermal properties as a function of the thermal properties of the individual layers are obtained.

It has been demonstrated that for thermally thick layers, the hyperbolic effects are more remarkable when the thermal relaxation time is close to the thermalization time of the medium. A new formula for the effective thermal diffusivity, in the hyperbolic approach, was obtained when both composing layers are thermally thick. It was also shown that the developed effective model provides a good approximation for the temperature profile of a layered system obeying the Cattaneo–Vernotte equation. The enhancement of heat transport is due to the fact that the analysis of the effective thermal diffusion privileges the role of thermal diffusivity above the thermal relaxation time. As a consequence, the spatial decay of the thermal wave when going through the material, considering hyperbolic effects, is smaller due to the effect of the second order time derivative in hyperbolic heat conduction equation. This induces an increase of effective thermal diffusivity of the composed material. Our results establish the basis for the development of hyperbolic heat transport models in complex systems and their applicability in the interpretation and analysis of experimental data.

References

[1] S. Torquato, Random Heterogeneous Materials, Springer-Verlag, New York, 2001.

[2] H.S. Carslaw, J.C. Jaeger, Conduction of Heat in Solids, Second ed. Oxford University Press, London, 1959.

[3] W.P. Leung, A.C. Tam, Techniques of flash radiometry, J. Appl. Phys. 56 (1) (1984) 153–161.

[4] D.D. Joseph, L. Preziosi, Heat waves, Rev. Mod. Phys. 61 (1) (1989) 41–73.

[5] M.N. Ozisik, D.Y. Tzou, On the wave theory in heat conduction, ASME J. Heat Transfer 116 (3) (1994) 526–535.

[6] L. Wang, X. Zhou, X. Wei, Heat Conduction: Mathematical Models and Analytical Solutions, Springer-Verlag, Berlin Heidelberg, 2008.

[7] A. Vedavarz, S. Kumar, M.K. Moallemi, Significance of non-Fourier heat waves in conduction, ASME J. Heat Transfer 116 (1) (1994) 221–224.

[8] R.K. Sahoo, Propagation of thermal waves with lateral heat transfer, Cryogenics 34 (3) (1994) 203–212.

[9] C. Cattaneo, Sulla coduzione del calore, Atti. Semin. Mat. Fis. Univ. Modena 3 (1948) 83.

[10] P. Vernotte, Les Paradoxes de la Theorie Continue de l'equation de la Chaleur, C.R. Acad. Sci. 246 (22) (1958) 3154–3155.

[11] S. Galovic, D. Kotoski, Photothermal wave propagation in media with thermal memory, J. Appl. Phys. 93 (5) (2003) 3063–3070.

[12] W. Kaminski, Hyperbolic heat conduction equation for materials with a nonhomogeneous inner structure, ASME J. Heat Transfer 112 (3) (1990) 555–560.

[13] K. Mitra, S. Kumar, A. Vedavarz, M.K. Moallemi, Experimental evidence of hyperbolic heat conduction in processed meat, ASME J. Heat Transfer 117 (3) (1995) 568–573.

[14] W. Roetzel, N. Putra, S.K. Das, Experiment and analysis for non-Fourier conduction in materials with non-homogeneous inner structure, Int. J. Thermal Sci. 42 (6) (2003) 541–552.

[15] D.Y. Tzou, A unified field approach for heat conduction from macro- to microscales, J. Heat Transfer 117 (1) (1995) 8–16.

[16] L. Cheng, M. Xu, L. Wang, From Boltzmann transport equation to single-phase-lagging heat conduction, Int. J. Heat Mass Transfer 51 (2008) 6018–6023.

[17] F. Xu, K.A. Seffen, T.J. Lu, Non-Fourier analysis of skin biothermomechanics, Int. J. Heat Mass Transfer 51 (9) (2008) 2237–2259.

[18] A. Graßmann, F. Peters, Experimental investigation of heat conduction in wet sand, Heat Mass Transfer 35 (4) (1999) 568–573.

[19] S.U.S Choi, Enhancing thermal conductivity of fluids with nanoparticles, Developments and Applications of Non-Newtonian Flows, FED-vol.231/MD-vol. 66, 1995, pp. 99–105.

[20] J.A. Eastman, S.U.S. Choi, S. Li, W. Yu, L.J. Thompson, Anomalously increased effective thermal conductivities of ethylene glycol-based nanofluids containing copper nanoparticles, Appl. Phys. Lett. 78 (6) (2001) 718–720.

[21] J.J. Vadasz, S. Govender, P. Vadasz, Heat transfer enhancement in nano-fluids suspensions: possible mechanisms and explanations, Int. J. Heat Mass Transfer 48 (13) (2005) 2673–2683.

[22] L. Wang, X. Wei, Heat conduction in nanofluids, Chaos, Solitons Fractals 39 (5) (2009) 2211–2215.

[23] D.P. Almond, P.M. Patel, Photothermal Science and Techniques, Chapman & Hall, London, 1996.

[24] T. Tominaga, K. Ito, Theory of photoacoustic measurements of the thermal diffusivity of two-layers samples, Japan. J. Appl. Phys. 27 (12) (1988) 2392–2397.

[25] A.M. Mansanares, A.C. Bento, H. Vargas, N.F. Leite, L.C.M. Miranda, Photoacustic measurement of the thermal properties of two-layers system, Phys. Rev. B 42 (7) (1990) 4477–4486.

[26] J.L. Lucio, J.J. Alvarado-Gil, O. Zelaya-Angel, H. Vargas, On the thermal properties of a two-layer system, Phys. Stat. Sol. A 150 (2) (1995) 695–704.

[27] E. Marin, J.L. Pichardo, A. Cruz-Orea, P. Díaz, G. Torres-Delgado, I. Delgadillo, J.J. Alvarado-Gil, J.G. Mendoza-Alvarez, H. Vargas, On the thermal characterization of two-layer systems by means of the photoacoustic effect, J. Phys. D: Appl. Phys. 29 (4) (1996) 981–986.

[28] A. Salazar, A. Sánchez-Lavega, J.M. Terrón, Effective thermal diffusivity of layered materials measured by modulated photothermal techniques, J. Appl. Phys. 84 (6) (1998) 3031–3041.
[29] J.I. Frankel, B. Vick, M.N. Özisik, General formulation and analysis of hyperbolic heat conduction in composite media, Int. J. Heat Mass Transfer 30 (7) (1987) 1293–1305.
[30] K. Ramadan, Semi-analytical solutions for the dual phase lag heat conduction in multilayered media, Int. J. Thermal Sci. 48 (1) (2009) 14–25.
[31] T. Niu, W. Dai, A hyperbolic two-step model based finite difference scheme for studying thermal deformation in a double-layered thin film exposed to ultrashort-pulsed lasers, Int. J. Thermal Sci. 48 (1) (2009) 34–49.
[32] Jeng-Rong Ho, Chun-Pao Kuo, Wen-Shu Jiaung, Study of heat transfer in multilayered structure within the framework of dual-phase-lag heat conduction model using lattice Boltzmann method, Int. J. Heat Mass Transfer 46 (1) (2003) 55–69.
[33] J. Ordóñez-Miranda, J.J. Alvarado-Gil, Thermal wave oscillations and thermal relaxation time determination in a hyperbolic heat transport model, doi:10.1016/j.ijthermalsci.2009.03.008.
[34] F. Alan Mcdonald, G.C. Westel Jr., Generalized theory of the photoacoustic effect, J. Appl. Phys. 49 (4) (1978) 2313–2322.
[35] A. Rosencwaig, A. Gersho, Theory of the photoacoustic effect with solids, J. Appl. Phys. 47 (1) (1976) 64–69.
[36] A.F. Khadrawi, M.A. Al-Nimr, M. Hammad, Thermal behavior of perfect and imperfect contact composite slab under the effect of the hyperbolic heat conduction model, Int. J. Thermophys. 23 (2) (2002) 581–598.
[37] J.L. Pichardo, J.J. Alvarado-Gil, Open photoacoustic cell determination of the thermal interface resistance in two layer systems, J. Appl. Phys. 89 (7) (2001) 4070–4075.
[38] A.M. Mansanares, H. Vargas, F. Galembeck, J. Buijs, D. Bicanic, Photoacoustic characterization of a two-layer system, J. Appl. Phys. 70 (11) (1991) 7046–7050.
[39] M.D. Dramicanin, Z.D. Ristovski, V. Djokovic, S. Galovic, Conduction of heat in inhomogeneous solids, Appl. Phys. Lett. 73 (3) (1998) 321–323.
[40] John P. Mckelvey, Solid State and Semiconductor Physics, Harper & Row, New York, 1966.
[41] J.J. Alvarado-Gil, O. Zelaya-Angel, H. Vargas, J.L. Lucio, Photoacoustic characterization of the thermal properties of a semiconductor-glass two-layer system, Phys. Rev. B 50 (19) (1994) 14627–14630.
[42] P. Vadasz, Absence of oscillations and resonance in porous media dual--phase-lagging Fourier heat conduction, ASME J. Heat Transfer 127 (3) (2005) 307–314.
[43] R. Quintanilla, R. Racke, A note on stability in three-phase-lag heat conduction, Int. J. Heat Mass Transfer 51 (1) (2008) 24–29.
[44] M.A. Al-Nimr, M. Naji, R.I. Abdallah, Thermal behavior of a multi-layered thin slab carrying pulsating signals under the effect of the dual-phase-lag heat conduction model, Int. J. Thermophys. 25 (3) (2004) 949–965.
[45] K. Ramadan, M.A. Al-Nimr, Analysis of the thermal behavior of a multilayer slab with imperfect contact using the dual phase lag heat conduction model, ASME J. Heat Transfer 130 (7) (2008) 074501–074505.
[46] N.S. Al-Huniti, M.A. Al-Nimr, Dynamic thermoelastic response of a heated thin composite plate using the hyperbolic heat conduction model, Int. J. Heat Technol. 22 (1) (2004) 179–189.
[47] M.A. Al-Nimr, V.S. Arpaci, The thermal behavior of thin metal films in the hyperbolic two-step model, Int. J. Heat Mass Transfer 43 (11) (2000) 2021–2028.
[48] M. Naji, M. Al-Nimr, T. Darabseh, Thermal stress investigation in unidirectional composites under the hyperbolic energy model, Int. J. Solids Struct. 44 (16) (2007) 5111–5121.
[49] N.S. Al-Huniti, M.A. Al-Nimr, Thermoelastic response of a heated thin composite plate using the hyperbolic heat conduction model: lumped analysis, Int. J. Thermal Sci. 43 (10) (2004) 959–965.